THE DIMINISHED BASE LOCUS IS NOT ALWAYS CLOSED

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Abstract. We exhibit a pseudoeffective $\mathbb{R}$-divisor D_λ on the blow-up of $\mathbb{P}^3$ at nine very general points which lies in the closed movable cone and has negative intersections with a set of curves whose union is Zariski dense. It follows that the diminished base locus $\mathbf{B}_-(D_\lambda) = \bigcup_{A \text{ ample}} \mathbf{B}(D_\lambda + A)$ is not closed and that D_λ does not admit a Zariski decomposition in even a very weak sense. By a similar method, we construct an $\mathbb{R}$-divisor on the family of blow-ups of $\mathbb{P}^2$ at ten distinct points, which is nef on a very general fiber but fails to be nef over countably many prime divisors in the base.

1. Introduction

For a pseudoeffective $\mathbb{R}$-divisor D on a normal projective variety Y, the diminished base locus (also called the non-nef locus or restricted base locus) is the union

$$\mathbf{B}_-(D) = \bigcup_{\substack{A \text{ ample} \\ D+A\ \mathbb{Q}\text{-Cartier}}} \mathbf{B}(D+A),$$

where $\mathbf{B}(D+A) = \bigcap_{n\geq 1} \mathrm{Bs}(n(D+A))$ is the stable base locus [8]. This is at most a countable union of subvarieties, but in many examples the union is finite, i.e. Zariski closed. We will give an example of an $\mathbb{R}$-divisor for which this locus is not Zariski closed.

Theorem 1.1. *Let X be the blow-up of $\mathbb{P}^3$ at nine very general points. There exists a pseudoeffective $\mathbb{R}$-divisor D_λ on X with the following properties:*

(1) *There is a countable set of curves $C_n \subset X$ with $D_\lambda \cdot C_n < 0$, whose union is Zariski dense on X.*
(2) *$\mathbf{B}_-(D_\lambda)$ is a countable union of curves.*
(3) *There is no decomposition $f^*D_\lambda \equiv_{\mathrm{num}} P + N$ of f^*D_λ into nef and effective components on any birational model $f : Y \to X$.*

Further, there exists a big $\mathbb{R}$-divisor D'_λ on $\mathbb{P}_X(\mathcal{O}_X \oplus \mathcal{O}_X(1))$ for which $\mathbf{B}_-(D'_\lambda)$ is a countable union of curves, where $\mathcal{O}_X(1)$ is any very ample line bundle on X.

A similar method gives an example related to the behavior of nefness of divisors in families.

Theorem 1.2. *Let $\Sigma = ((\mathbb{P}^2)^{10} \backslash \Delta)/\operatorname{PGL}(3)$, where Δ is the locus where two points coincide, and let $\mathcal{X} \to \Sigma$ be the family whose fiber over $\mathbf{p} \in \Sigma$ is isomorphic to the blow-up of $\mathbb{P}^2$ at the corresponding ten points. There exists an $\mathbb{R}$-divisor C_λ on $\mathcal{X}$ such that $C_{\lambda,\mathbf{p}}$ is nef for very general $\mathbf{p}$, but there are countably many prime divisors $V_n \subset \Sigma$ such that $C_{\lambda,\mathbf{p}}$ is not nef if $\mathbf{p} \in V_n$.*

The behavior of this example is an instance of the following property of nefness.

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Proposition ([12], Proposition 1.4.14)**.** Suppose that X and S are varieties over a field and $\pi : X \to S$ is a surjective and proper morphism. Let D be an $\mathbb{R}$-Cartier divisor on X. If D_0 is nef for some $0 \in S$, then D_s is nef for very general $s \in S$ (i.e. for all s not contained in some countable union of subvarieties).

There do not seem to be any examples known in characteristic 0 in which D is a Cartier divisor and nefness is not simply an open condition. The example demonstrates that, at least in the generality of $\mathbb{R}$-divisors, the "very general" of the conclusion is indeed essential. Some recent examples in positive and mixed characteristic are discussed in [11].

Both examples arise from classical constructions. Throughout, we work over an uncountable algebraically closed field of arbitrary characteristic. Starting with a set of k very general points on $\mathbb{P}^n$, and a divisor of degree d with given multiplicities at these points, we make a sequence of Cremona transformations centered at certain subsets of the points, and compute the degree and multiplicities of the strict transform of the divisor under these transformations. The changes in degrees and multiplicities are governed by an action of a Coxeter group of type $T_{2,n+1,k-n-1}$, an observation originally due to Coble [6]. A modern account of Coble's work can be found in the survey of Dolgachev and Ortland [7]. If $n = 2$ and $k = 10$, or $n = 3$ and $k = 9$, then the associated groups of type $T_{2,3,7}$ and $T_{2,4,5}$ are infinite and have elements acting with eigenvalues of norm greater than 1. The divisors of the examples arise as the corresponding eigenvectors. For simplicity, we make explicit computations for specific transformations in these groups, but the results are valid for many other elements as well.

These Coxeter groups have often played a role in the study of the birational geometry of blow-ups of projective space. Nagata's construction of infinitely many (-1)-curves on blow-ups of $\mathbb{P}^2$ at 9 very general points makes use of the fact that the group of type $T_{2,3,6}$ is infinite [15], while Mukai's characterization of the blow-ups of $\mathbb{P}^n$ which are Mori dream spaces again relies on the finiteness of associated Coxeter groups [14]. Laface and Ugaglia's study of a higher-dimensional analogue of the Harbourne-Hirschowitz conjecture also involves sequences of Cremona transformations centered at various points [9].

Elements of these groups have been studied from a dynamical perspective as well: for blow-ups of $\mathbb{P}^n$ at special configurations of points, there can exist (pseudo-)automorphisms whose action on cohomology is given by these elements. This was studied in dimension 2 in the work of McMullen [13] and Bedford-Kim [2],[3], and in higher dimension by Perroni-Zhang [17]. Eigenvectors of the action on cohomology of the automorphisms in [13] provide examples of the sort in Theorem 1.2. I have learned that recent work of T. Bayraktar also considers the diminished base loci of a general class of $\mathbb{R}$-divisors constructed as eigenvectors of pseudo-automorphisms; the example presented here is roughly one in which the inclusion in Theorem 1.1 of [1] is an equality, and the union is infinite.

The next section contains some preliminary lemmas needed for the constructions. Section 3 provides the example of Theorem 1.2. Section 4 introduces the standard Cremona transformation $\mathrm{Cr} : \mathbb{P}^3 \dashrightarrow \mathbb{P}^3$, leading to the construction of D_λ. The various claims of Theorem 1.1 are proved in Sections 5 and 6 as Lemmas 5.2, 6.3, and 5.4.

2. Preliminaries

We first record a simple observation which implies that $\mathbb{R}$-divisors arising as eigenvectors of automorphisms of $N^1(X)$ often generate extremal rays on the various cones of divisors.

Lemma 2.1 (cf. [5])**.** *Suppose that V is a finite dimensional real vector space, $G \subset V$ is a closed convex cone with nonempty interior and containing no line, and $T : V \to V$ is a linear map with $T(G) = G$. If T has a real eigenvalue λ of algebraic multiplicity one, with magnitude larger than that of any other eigenvalue, then the λ-eigenvector v_λ (with appropriate sign) spans an extremal ray on G.*

Proof. Fix a norm $|\cdot|$ on V and write $V = \mathbb{R}v_\lambda \oplus W$, where W is the direct sum of the other real Jordan blocks, so that $T|_W$ has all eigenvalues with norm strictly less than λ. Since G has nonempty interior, there exists $v \in G$ with nonzero component in the v_λ-eigenspace. Then $\frac{1}{\lambda^n}T^n v$ converges to some nonzero multiple of v_λ. Switching the sign if needed, we conclude that v_λ is contained in G.

Suppose that v_λ is not extremal, i.e. that there exists a nonzero $w \in W$ for which $v_\lambda + w$ and $v_\lambda - w$ are both in G. Since its image contains an open set, T is invertible and $T^{-1}(G) = G$. There is a sequence n_i for which $T^{-n_i}w/\left|T^{-n_i}w\right|$ converges to a nonzero limit $r \in V$. Since $T|_W$ has eigenvalues less than λ, $|\lambda^n T^{-n} w|$ grows without bound as n increases, and $v_\lambda/\left|\lambda^n T^{-n}w\right|$ converges to 0. It follows that the two sequences of vectors in G

$$\frac{\lambda^{n_i}T^{-n_i}(v_\lambda \pm w)}{|\lambda^{n_i}T^{-n_i}w|} = \frac{v_\lambda}{|\lambda^{n_i}T^{-n_i}w|} \pm \frac{\lambda^{n_i}T^{-n_i}w}{|\lambda^{n_i}T^{-n_i}w|}$$

converge to $\pm r$. The closedness of G implies that both r and $-r$ are contained in G, contradicting the assumption that G contains no line. □

Both examples deal with blow-ups of projective space, and it will be useful to establish some basic notation for k-tuples of points on $\mathbb{P}^n$. Throughout, we work over an uncountable algebraically closed field of arbitrary characteristic. Let $\Sigma = ((\mathbb{P}^n)^k \setminus \Delta)/\operatorname{PGL}(n+1)$ be the set of k-tuples with all points distinct, and let $\pi : \mathcal{X} \to \Sigma$ be the family whose fiber $X_{\mathbf{p}}$ over $\mathbf{p} = (p_1, \ldots, p_k) \in \Sigma$ is isomorphic to the blow-up of $\mathbb{P}^n$ at the corresponding k points.

If Y is a normal projective variety, the group of $\mathbb{R}$-Cartier divisors on Y modulo numerical equivalence is denoted $N^1(Y)$, and $[D] \in N^1(Y)$ is the numerical class of a divisor D, though when no confusion is possible we omit the brackets. Dually, $N_1(Y)$ is the group of curves modulo numerical equivalence, and the class of C is written $[C]$.

For any fiber $X = X_{\mathbf{p}}$, there are decompositions $N^1(X) = \mathbb{R}H \oplus \bigoplus_{i=1}^k \mathbb{R}E_i$ and $N_1(X) = \mathbb{R}h \oplus \bigoplus_{i=1}^k \mathbb{R}e_i$, where H is the pullback of the hyperplane class on $\mathbb{P}^n$, h is the class of the strict transform of a line disjoint from the points of $\mathbf{p}$, E_i are the exceptional divisors, and e_i the classes of lines in the E_i. We will refer to $H, E_1, \ldots, E_k$ and $h, e_1, \ldots, e_k$ as the standard bases for $N^1(X)$ and $N_1(X)$. The intersection pairing on the exceptional classes is given by $E_i \cdot e_i = -1$. If $\mathbf{p}$ and $\mathbf{q}$ are two different sets of points, there is an isomorphism $\Phi_{\mathbf{pq}} : N^1(X_{\mathbf{p}}) \to N^1(X_{\mathbf{q}})$ which sends $H_{\mathbf{p}}$ to $H_{\mathbf{q}}$ and $E_{i,\mathbf{p}}$ to $E_{i,\mathbf{q}}$; the matrix for $\Phi_{\mathbf{pq}}$ with respect to the above bases is the $(k+1) \times (k+1)$ identity matrix.

The pseudoeffective cone $\overline{\operatorname{Eff}}(X) \subset N^1(X)$ is the closure of the cone $\operatorname{Eff}(X)$ generated by classes of effective Cartier divisors, and the movable cone $\overline{\operatorname{Mov}}(X) \subset N^1(X)$ is the closure of the cone generated by classes of such divisors whose base locus has codimension at least 2. The next lemma shows that if $\mathbf{p}$ and $\mathbf{q}$ are very general, the movable and pseudoeffective cones of $X_{\mathbf{p}}$ and $X_{\mathbf{q}}$ coincide under the identification $\Phi_{\mathbf{pq}}$.

Lemma 2.2. *There is a set $U \subset \Sigma$, the complement of a countable union of subvarieties, such that if $\mathbf{p}$ and $\mathbf{q}$ lie in U, then $\Phi_{\mathbf{pq}}(\operatorname{Eff}(X_{\mathbf{p}})) = \operatorname{Eff}(X_{\mathbf{q}})$, $\Phi_{\mathbf{pq}}(\overline{\operatorname{Eff}}(X_{\mathbf{p}})) = \overline{\operatorname{Eff}}(X_{\mathbf{q}})$, and $\Phi_{\mathbf{pq}}(\overline{\operatorname{Mov}}(X_{\mathbf{p}})) = \overline{\operatorname{Mov}}(X_{\mathbf{q}})$.*

Proof. Since every divisor class on $X_{\mathbf{p}}$ or $X_{\mathbf{q}}$ is the restriction of a class on $\mathcal{X}$, to check that the effective (resp. movable) cones coincide, it is enough to check that for very general $\mathbf{p}$ and $\mathbf{q}$, precisely the same integral classes D on $\mathcal{X}$ have effective (resp. movable) restrictions to $X_{\mathbf{p}}$ and $X_{\mathbf{q}}$.

For a given class $D = dH - \sum m_i E_i$ on $\mathcal{X}$, the set of $\mathbf{p}$ for which $h^0(X_{\mathbf{p}}, D_{\mathbf{p}}) > 0$ is a closed subset of Σ by the semicontinuity theorem. It follows that $X_{\mathbf{p}}$ and $X_{\mathbf{q}}$ have the same effective integral classes as long as these two points lie off of the countably many proper closed subsets that arise in this way, and so the effective and pseudoeffective cones coincide.

For the movable cone, we again restrict our attention to integral classes. An integral class $D_{\mathbf{p}}$ on $X_{\mathbf{p}}$ has base locus of codimension 2 if $h^0(X_{\mathbf{p}}, D_{\mathbf{p}}) > 1$ and $D_{\mathbf{p}}$ has no fixed part, i.e. there does not exist a nonzero class $F_{\mathbf{p}}$ with $h^0(X_{\mathbf{p}}, F_{\mathbf{p}}) > 0$ such that $h^0(X_{\mathbf{p}}, D_{\mathbf{p}} - F_{\mathbf{p}}) = h^0(X_{\mathbf{p}}, D_{\mathbf{p}})$. The result follows as above by the fact that for any integral D and F on X, each of $h^0(X_{\mathbf{p}}, F_{\mathbf{p}})$, $h^0(X_{\mathbf{p}}, D_{\mathbf{p}} - F_{\mathbf{p}})$, and $h^0(X_{\mathbf{p}}, D_{\mathbf{p}})$ is constant for $\mathbf{p}$ in some open set, and only countably many D and F are considered. □

We will use the following properties of the diminished base locus, which follow from the definition (see [8] for details).

Lemma 2.3. *Suppose that D is a pseudoeffective $\mathbb{R}$-divisor on a normal projective variety Y.*

(1) $\mathbf{B}_-(D)$ *depends only on the numerical class* $[D] \in N^1(Y)$.
(2) $\mathbf{B}_-(D) = \emptyset$ *if and only if D is nef.*
(3) *If C is a curve with $D \cdot C < 0$, then $C \subset \mathbf{B}_-(D)$.*
(4) $\mathbf{B}_-(D + D') \subseteq \mathbf{B}_-(D) \cup \mathbf{B}_-(D')$.
(5) *If $f : Y' \to Y$ is a surjective morphism between smooth varieties, $\mathbf{B}_-(f^*D) = f^{-1}(\mathbf{B}_-(D))$.*
(6) *If $\{A_i\}$ is a sequence of ample divisors converging to 0 in $N^1(Y)$, with each $D + A_i$ a $\mathbb{Q}$-divisor, then $\mathbf{B}_-(D) = \bigcup_j \mathbf{B}(D + A_j)$.*
(7) *If $D \in \overline{\mathrm{Mov}}(X)$, then every component of $\mathbf{B}_-(D)$ has codimension at least 2.*

3. Nefness in families of $\mathbb{R}$-divisors

In this section, we adopt the notation of Section 2 for blow-ups of $\mathbb{P}^2$ at $k = 10$ points. The proof of Theorem 1.2 is contained in Lemmas 3.1, 3.3, and 3.4.

The standard Cremona transformation $\mathrm{Cr} : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ given by $[X_1, X_2, X_3] \mapsto [X_1^{-1}, X_2^{-1}, X_3^{-1}]$ has a resolution

$$\begin{array}{ccc} X & = & X' \\ {\scriptstyle \pi}\downarrow & & \downarrow{\scriptstyle \pi'} \\ \mathbb{P}^2 & \overset{\mathrm{Cr}}{\dashrightarrow} & \mathbb{P}^2 \end{array}$$

Here π is the blow-up of $\mathbb{P}^2$ at three points, and π' contracts the strict transforms of the lines between any two of those points. We employ the two notations X and X' to emphasize that the standard bases $\{h, e_1, e_2, e_3\}$ and $\{h', e_1', e_2', e_3'\}$ are different. If C is any curve on

X, then its class on X' in the new basis is given by $M([C])$ where

$$M = \begin{pmatrix} 2 & 1 & 1 & 1 \\ -1 & 0 & -1 & -1 \\ -1 & -1 & 0 & -1 \\ -1 & -1 & -1 & 0 \end{pmatrix}.$$

If $\mathbf{p} \in \Sigma$ is a configuration with p_8, p_9, and p_{10} in linear general position, there is a Cremona transformation $\mathrm{Cr}_{\mathbf{p}} : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ defined by $g^{-1} \circ \mathrm{Cr} \circ g$, where g an element of $\mathrm{Aut}(\mathbb{P}^2)$ sending p_8, p_9, p_{10} to the points $[1,0,0]$, $[0,1,0]$, $[0,0,1]$, and Cr is the standard Cremona transformation. Let $\rho : \Sigma \dashrightarrow \Sigma$ be the rational map given by $(p_1,\ldots,p_{10}) \mapsto (p_8, p_9, p_{10}, \mathrm{Cr}_{\mathbf{p}}(p_1),\ldots,\mathrm{Cr}_{\mathbf{p}}(p_7))$. This map is regular off of the set $L \subset \Sigma$, defined as the locus of $\mathbf{p}$ with some p_i on a line through two of p_8, p_9, and p_{10}. Let $\mathbf{p}$ be any point of $\Sigma \setminus L$, and set $\mathbf{q} = \rho(\mathbf{p})$. Write Π_σ for the permutation matrix for $\sigma = (8,9,10,1,2,3,4,5,6,7)$, and consider the map $M_\sigma^{\mathbf{pq}} : N^1(X_{\mathbf{p}}) \to N^1(X_{\mathbf{q}})$ given in the standard bases by

$$M_\sigma^{\mathbf{pq}} = \left(\begin{array}{c|c} M & 0 \\ \hline 0 & I_7 \end{array}\right)\left(\begin{array}{c|c} 1 & 0 \\ \hline 0 & \Pi_\sigma \end{array}\right),$$

where both of these are 11×11 block matrices, but with different block sizes. If C is a curve on $X_{\mathbf{p}}$, there is a curve on $X_{\mathbf{q}}$ lying in the class $M_\sigma^{\mathbf{pq}}([C])$, obtained by cyclically reordering the points so the last three come first, and then taking the strict transform of C under a Cremona transformation centered at these points. If $\mathbf{p}$ is in very general position, then the map $\Phi_{\mathbf{pq}}^{-1} : N^1(X_{\mathbf{q}}) \to N^1(X_{\mathbf{p}})$ is an isomorphism which identifies the effective cones, by Lemma 2.2. Let $M_\sigma = \Phi_{\mathbf{pq}}^{-1} \circ M_\sigma^{\mathbf{pq}} : N^1(X_{\mathbf{p}}) \to N^1(X_{\mathbf{p}})$ be the composition.

For very general $\mathbf{p}$, since both $\Phi_{\mathbf{pq}}^{-1}$ and $M_\sigma^{\mathbf{pq}}$ identify the effective cones, we have $M_\sigma(\overline{\mathrm{Eff}}(X_{\mathbf{p}})) = \overline{\mathrm{Eff}}(X_{\mathbf{p}})$. Because M_σ preserves the intersection form on $N^1(X_{\mathbf{p}})$, it satisfies $M_\sigma(\mathrm{Nef}(X_{\mathbf{p}})) = \mathrm{Nef}(X_{\mathbf{p}})$ as well.

If $\mathbf{p}$ is a point for which the effective cone of $X_{\mathbf{p}}$ is larger than that of a very general configuration, it is not necessarily the case that $\Phi_{\mathbf{pq}}^{-1}$ maps effective classes to effective classes. The divisor of the example will fail to be nef precisely over certain configurations $\mathbf{p}$ with the first three points collinear, the first six on a conic, etc. These are “nodal relations” among the points, in the terminology of McMullen [13].

Lemma 3.1. *The map M_σ has characteristic polynomial $(t-1)t^5 q(t+t^{-1})$, where $q(t) = t^5 - t^4 - 6t^3 + 5t^2 + 8t - 5$. M_σ has a unique eigenvalue $\lambda \approx 1.431$ of magnitude greater than 1. When the λ-eigenvector $C_{\lambda,\mathbf{p}}$ is written as $h - \sum_{i=1}^{10} r_i e_i$, the first three coefficients satisfy $r_1 + r_2 + r_3 > 1$.*

The divisor $C_{\lambda,\mathbf{p}}$ is nef on $X_{\mathbf{p}}$ for very general $\mathbf{p}$.

Proof. The inequality on the coefficients can be checked by computing an approximation of the eigenvalue and then expressing each of the coefficients as a rational function of λ. The claimed nefness then follows from Lemma 2.1, with the cone $G = \mathrm{Nef}(X_{\mathbf{p}}) \subset N^1(X)$ and with M_σ for the linear map T. □

Remark 3.2. In the standard coordinates, the divisor is approximately

$$C_\lambda \approx (1, -0.451, -0.440, -0.408, -0.315, -0.307, -0.285, -0.220, -0.215, -0.199, -0.154).$$

Let C_λ be the corresponding divisor $h - \sum_{i=1}^{10} r_i e_i$ on the total space $\mathcal{X}$. Though $C_{\lambda,\mathbf{p}}$ is nef for very general $\mathbf{p}$, we will see that if $\mathbf{p}$ lies on any of countably many subvarieties V_n

of Σ for which $X_{\mathbf{p}}$ contains (-2)-curves of certain classes, $C_{\lambda,\mathbf{p}}$ is not nef. Define V_0 to be the set of $\mathbf{p} \in \Sigma$ for which p_1, p_2, and p_3 are collinear. If $\mathbf{p}_0 \in V_0$, there is a curve $\bar{\ell} \subset X_{\mathbf{p}_0}$ of class $C_0 = h - e_1 - e_2 - e_3$. Then $C_{\lambda,\mathbf{p}_0} \cdot \bar{\ell} = 1 - r_1 - r_2 - r_3 < 0$, and $C_{\lambda,\mathbf{p}_0}$ is not nef. Similarly, $\mathbf{p}_1 = \rho(\mathbf{p}_0)$ is a configuration of points with the first six lying on a conic, and the strict transform of that conic on $X_{\mathbf{p}_1}$ has negative intersection with $C_{\lambda,\mathbf{p}_1}$. Generally, for $n \geq 0$ define $V_{n+1} \subset \Sigma$ to be the strict transform of V_n under ρ.

Lemma 3.3. *Each V_n is a prime divisor not equal to L, and V_m and V_n are distinct if $m \neq n$. For any point $\mathbf{p}_n \in V_n$, there exists a curve $C_n \subset X_{\mathbf{p}_n}$ in the class $M_\sigma^n(C_0)$.*

Proof. To prove these sets are distinct, we will construct a sequence of points $\mathbf{p}_n \in V_n \setminus L$ such that $X_{\mathbf{p}_n}$ contains a curve lying in the class $M_\sigma^n(C_0)$, which is the unique rational curve of self-intersection less than or equal -2. Let $E \subset \mathbb{P}^2$ be a smooth elliptic curve. Construct $\mathbf{p}_0 \in V_0$ by choosing points on E such that p_1, p_2, and p_3 are the points of intersection of E with some line ℓ meeting E transversely, and $p_4, \dots, p_{10}$ have the property that if $3d - \sum_{i=1}^{10} m_i = 0$, the class $d\ell|_E - \sum_{i=1}^{10} m_i p_i$ is not linearly equivalent to 0 on E unless $m_4 = \cdots = m_{10} = 0$. This condition will be met if these points are chosen to be very general. Write $\bar{\ell}$ and $\bar{E}$ for the strict transforms of ℓ and E on $X_{\mathbf{p}_0}$.

Suppose that $C \sim d\pi^* h - \sum_{i=1}^{10} m_i e_i$ is a rational curve with $K_{X_{\mathbf{p}_0}} \cdot C \geq 0$. Since $K_{X_{\mathbf{p}_0}} \sim -\bar{E}$, we have $\bar{E} \cdot C \leq 0$, and so $\bar{E} \cdot C = 0$. Then $3d - \sum_{i=1}^{10} m_i = 0$, and the hypothesis on the points implies that $C \sim h - e_1 - e_2 - e_3$ is the curve $\bar{\ell}$. It follows that under any sequence of Cremona transformations, no three points will become collinear; indeed, the strict transform of a line containing these three points would be a (-2)-curve on $X_{\mathbf{p}_0}$, but $\bar{\ell}$ is the only such. We may therefore define a sequence of points $\mathbf{p}_{n+1} \in V_{n+1}$ by taking $\mathbf{p}_{n+1} = \rho(\mathbf{p}_n)$. For each n, the image C_n of $\bar{\ell}$ is the unique (-2)-curve on $X_{\mathbf{p}_n}$, and lies in the class $M_\sigma^n(C_0)$, where $C_0 = h - e_1 - e_2 - e_3$. This implies that the divisors V_m are distinct.

A general point $\mathbf{p}_n \in V_n$ is of the form $\rho^n(\mathbf{p}_0)$ for some point $\mathbf{p}_0 \in V_0$. The strict transform on $X_{\mathbf{p}_0}$ of a line through the first three points of $\mathbf{p}_0$ has class C_0, and this curve has class $M_\sigma^n(C_0)$ on $X_{\mathbf{p}_n}$. □

Lemma 3.4. *If $\mathbf{p} \in V_n$, then $C_{\lambda,\mathbf{p}}$ is not nef.*

Proof. For any point $\mathbf{p} \in V_n$, there is a curve $C_n \subset X_{\mathbf{p}}$ with class $M_\sigma^n(C_0)$. Then

$$C_{\lambda,\mathbf{p}} \cdot C_n = \left(\frac{1}{\lambda^n} M_\sigma^n([C_{\lambda,\mathbf{p}}]) \right) \cdot M_\sigma^n([C_0]) = \frac{1}{\lambda^n} [C_{\lambda,\mathbf{p}}] \cdot [C_0] < 0. \qquad \square$$

Remark 3.5. The matrix $\left(\begin{smallmatrix} M & 0 \\ 0 & I_7 \end{smallmatrix}\right)$ corresponding to a Cremona transformation, together with the matrices $\left(\begin{smallmatrix} 1 & 0 \\ 0 & P \end{smallmatrix}\right)$ which permute the exceptional components by a permutation P, define an action of the Coxeter group of type $T_{2,3,7}$ on $N^1(X_{\mathbf{p}})$. This action is explored more fully in the original work of Coble [6] and the book of Dolgachev and Ortland [7]. Elements of this group correspond to a finite sequences of permutations of the points followed by Cremona transformations. Any such product preserves the nef and effective cones for very general $\mathbf{p}$, and so the construction carried out above works just as well for other elements of this group with a leading eigenvalue larger than 1.

4. The standard Cremona transformation and its iterates

We now turn to the second example, and will employ the notation of Section 2 for blow-ups of $\mathbb{P}^3$. Some notation from Section 3 will be reused in the new context. The example of

Theorem 1.1 will be constructed as an eigenvector of a map $N^1(X) \to N^1(X)$ induced on a blow-up of $\mathbb{P}^3$ by a certain sequence of Cremona transformations.

The standard Cremona transformation of $\mathbb{P}^3$ centered at four non-coplanar points $p_1, \dots, p_4$ is the birational map $\mathrm{Cr} : \mathbb{P}^3 \dashrightarrow \mathbb{P}^3$ defined by

$$[X_1, X_2, X_3, X_4] \mapsto [X_1^{-1}, X_2^{-1}, X_3^{-1}, X_4^{-1}],$$

where the coordinates are chosen so the points p_i lie at the intersections of the coordinate hyperplanes. The map Cr is toric and is easily seen to have a resolution

$$\begin{array}{ccccc} & & Y & & \\ & \swarrow^{p} & & \searrow^{p'} & \\ X & & \overset{\overline{\mathrm{Cr}}}{\dashrightarrow} & & X' \\ {\scriptstyle\pi}\downarrow & & & & \downarrow{\scriptstyle\pi'} \\ \mathbb{P}^3 & & \overset{\mathrm{Cr}}{\dashrightarrow} & & \mathbb{P}^3 \end{array}$$

Here both $\pi : X \to \mathbb{P}^3$ and $\pi' : X' \to \mathbb{P}^3$ are the blow-up of $\mathbb{P}^3$ at $p_1, \dots, p_4$, with exceptional divisors E_i and E'_i respectively. Let F_i and F'_i denote the strict transforms on X and X' of planes through the three points other than p_i, and H and H' the pullbacks of $\mathcal{O}_{\mathbb{P}^3}(1)$. Take l_{ij} and l'_{ij} to be the lines in $\mathbb{P}^3$ through p_i and p_j, and $\bar{l}_{ij}$ and $\bar{l}'_{ij}$ their strict transforms. The $\bar{l}_{ij}$ are smooth rational curves with normal bundle $\mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1)$ and $\overline{\mathrm{Cr}}$ is the flop of these six curves. More precisely, p is the blow-up of X along the six curves $\bar{l}_{ij}$, with exceptional divisors isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$, and these are contracted along the other ruling by p'. The strict transform of F_i under $\overline{\mathrm{Cr}}$ is the exceptional divisor E'_i, while the strict transform of E_i is F'_i.

The indeterminacy locus of $\overline{\mathrm{Cr}} : X \dashrightarrow X'$ is the union of the six curves $\bar{l}_{ij}$; since this map is an isomorphism in codimension 1, taking strict transforms of divisors induces an isomorphism $M : N^1(X) \to N^1(X')$, as well as an isomorphism $\check{M} : N_1(X) \to N_1(X')$ defined by requiring $D \cdot C = MD \cdot \check{M}C$. This action has been studied by Laface and Ugaglia in connection with special linear systems of divisors on $\mathbb{P}^3$ [9],[10]. In that context M describes the change in the multiplicity of a divisor at prescribed points under Cremona transformations centered at those points.

Lemma 4.1. *The isomorphisms $M : N^1(X) \to N^1(X')$ and $\check{M} : N_1(X) \to N_1(X')$ are given in the standard bases by the matrices*

$$M = \begin{pmatrix} 3 & 1 & 1 & 1 & 1 \\ -2 & 0 & -1 & -1 & -1 \\ -2 & -1 & 0 & -1 & -1 \\ -2 & -1 & -1 & 0 & -1 \\ -2 & -1 & -1 & -1 & 0 \end{pmatrix}, \quad \check{M} = \begin{pmatrix} 3 & 2 & 2 & 2 & 2 \\ -1 & 0 & -1 & -1 & -1 \\ -1 & -1 & 0 & -1 & -1 \\ -1 & -1 & -1 & 0 & -1 \\ -1 & -1 & -1 & -1 & 0 \end{pmatrix}.$$

Proof. For every i, we may write $H \sim F_i + \sum_{j \neq i} E_j$, and so $4H \sim \sum_{i=1}^4 F_i + 3\sum_{j=1}^4 E_j$. Similarly, $4H' \sim \sum_{i=1}^4 F'_i + 3\sum_{j=1}^4 E'_j$. Taking strict transforms yields $4M(H) = \sum_{i=1}^4 E'_i + 3\sum_{j=1}^4 F'_j$, and $4M(H) - 12H' = -8\sum_{j=1}^4 E'_i$. This gives $M(H) = 3H' - 2\sum_{j=1}^4 E'_j$, which is the first column of M. For the other columns, write $M(E_i) = F'_i = H' - \sum_{j \neq i} E'_j$. The matrix for $\check{M}$ is then determined by $M^t I_{1,4} \check{M} = I_{1,4}$, which $I_{1,4}$ is a 5×5 diagonal matrix with diagonal entries $(1, -1, -1, -1, -1)$. $\square$

If $\mathbf{p} \in \Sigma$ is a set of $k \geq 4$ points in linear general position (i.e. with no more than three coplanar), there is a Cremona transformation $\mathrm{Cr}_{\mathbf{p}} : \mathbb{P}^3 \dashrightarrow \mathbb{P}^3$ defined as $g^{-1} \circ \mathrm{Cr} \circ g$, where g is an automorphism $\mathbb{P}^3$ which sends $p_1, \ldots, p_4$ to the standard coordinate points. This is well-defined, since the parameter space Σ parametrizes points only up to automorphism. This Cremona transformation centered at the first four points induces a birational map which is an isomorphism in codimension 1, again denoted by $\overline{\mathrm{Cr}}_{\mathbf{p}} : X_{\mathbf{p}} \dashrightarrow X_{\mathbf{q}}$, where $\mathbf{q} = (p_1, \ldots, p_4, \mathrm{Cr}_{\mathbf{p}}(p_5), \ldots, \mathrm{Cr}_{\mathbf{p}}(p_k))$.

Corollary 4.2. *Suppose that* $\mathbf{p}$ *is a* k*-tuple in* $\mathbb{P}^3$ *with no four points coplanar and consider the map* $\overline{\mathrm{Cr}} : X_{\mathbf{p}} \dashrightarrow X_{\mathbf{q}}$ *induced by a standard Cremona transformation centered at the first four points.*

(1) *If* D *is any divisor on* $X_{\mathbf{p}}$*, then*

$$[\overline{\mathrm{Cr}}_{\mathbf{p}}(D)] = \left(\begin{array}{c|c} M & 0 \\ \hline 0 & I_{k-4} \end{array}\right)([D]),$$

where $\overline{\mathrm{Cr}}_{\mathbf{p}}(D)$ *denotes the strict transform of* D*.*

(2) *If* C *is any curve on* $X_{\mathbf{p}}$ *which does not meet the curves* $\bar{l}_{ij}$ *which make up the indeterminacy locus of* $\overline{\mathrm{Cr}}$*, then*

$$[\overline{\mathrm{Cr}}_{\mathbf{p}}(C)] = \left(\begin{array}{c|c} \check{M} & 0 \\ \hline 0 & I_{k-4} \end{array}\right)([C]),$$

where $\overline{\mathrm{Cr}}_{\mathbf{p}}(C)$ *denotes the strict transform of* C*.*

Proof. The strict transform of E_i is E_i' for $i > 4$, so the coefficients on these divisors are unaffected, and (1) is just Lemma 4.1. (2) follows from the fact that if C is disjoint from the indeterminacy locus of $\overline{\mathrm{Cr}}$, the intersection of C with a divisor is unchanged under strict transform, and $\begin{pmatrix} \check{M} & 0 \\ 0 & I_{k-4} \end{pmatrix}$ is the linear map which preserves the intersection form. $\square$

We now focus on the case that $k = 9$ points are blown up. If I is a 4-tuple from among the nine points, there is a birational map $\mathrm{Cr}_I : \mathbb{P}^3 \dashrightarrow \mathbb{P}^3$ defined as a standard Cremona transformation centered at the first four points of I, inducing a birational map $\overline{\mathrm{Cr}}_I : X_{\mathbf{p}} \to X_{\mathbf{q}}$ which is an isomorphism in codimension 1. Given a sequence $\mathbf{I} = (I_1, \ldots, I_n)$ of 4-tuples from among the nine points, the composition $\mathrm{Cr}_{\mathbf{I}} = \mathrm{Cr}_{I_n} \circ \cdots \circ \mathrm{Cr}_{I_1}$ is not defined in general; four of the points might become coplanar under some $\mathrm{Cr}_{I_{j-1}}$. However, if $\mathbf{p} = \mathbf{p}_0$ is in very general position, arbitrary compositions of Cremona transformations are defined. When the composition is defined, we write $\overline{\mathrm{Cr}}_{I_j} : X_{\mathbf{p}_{j-1}} \dashrightarrow X_{\mathbf{p}_j}$ for the induced birational maps of the blow-ups, and $\overline{\mathrm{Cr}}_{\mathbf{I}} : X_{\mathbf{p}_0} \dashrightarrow X_{\mathbf{p}_n}$ for their composition.

If $\bar{\ell} \subset X_{\mathbf{p}}$ is the strict transform of a line through p_1 and p_2, the numerical class of its strict transform under $\overline{\mathrm{Cr}}_{\mathbf{I}}$ could be computed using Corollary 4.2 if it were known that the strict transform of $\bar{\ell}$ under $\overline{\mathrm{Cr}}_{I_{k-1}} \circ \cdots \circ \overline{\mathrm{Cr}}_{I_1}$ is disjoint from the indeterminacy locus of $\overline{\mathrm{Cr}}_{I_k}$ for every $k \leq n-1$. Laface and Ugaglia have shown that this is indeed the case for very general blow-ups. The strategy of the proof is to specialize to the situation where the points lie on a genus 1 curve, and reduce the claimed disjointness to the nonvanishing of certain combinations of the points in the Picard group.

Theorem 4.3 ([10], Proposition 2.7)**.** *Let* $\mathbf{I} = (I_1, \ldots, I_n)$ *be a finite sequence of* 4*-tuples, and let* ℓ *be the line in* $\mathbb{P}^3$ *between* p_1 *and* p_2*, with* $\bar{\ell}$ *its strict transform on* $X = X_{\mathbf{p}}$*. There exists an open subset* $U_{\mathbf{I}} \subset \Sigma$ *such that if* $\mathbf{p}$ *is contained in* $U_{\mathbf{I}}$*, the following hold:*

(1) *The composition* $\mathrm{Cr}_{\mathbf{I}} : \mathbb{P}^3 \dashrightarrow \mathbb{P}^3$ *is well-defined.*
(2) *If* $\bar{\ell}$ *is not contained in the indeterminacy locus of* $\overline{\mathrm{Cr}}_{\mathbf{I}}$*, then for each* $1 \le j \le n$*, the strict transform* $\bar{\ell}_{j-1} \subset X_{\mathbf{p}_{j-1}}$ *is disjoint from the indeterminacy locus of* $\overline{\mathrm{Cr}}_{I_j}$*.*

We now consider compositions of Cremona transformations centered at judiciously chosen sequences of quadruples from the among nine points. Let $\sigma \in S_9$ be the permutation $(6, 7, 8, 9, 1, 2, 3, 4, 5)$, and take $I_j = (\sigma^{-j}(1), \dots, \sigma^{-j}(4))$. The composition $\mathrm{Cr}_{I_j} \circ \cdots \circ \mathrm{Cr}_{I_1}$ could equivalently be realized by repeatedly making a Cremona transformation centered at $p_6, \dots, p_9$ and then cyclically permuting the indices so these points become $p_1, \dots, p_4$.

Let $X = X_{\mathbf{p}}$ be the blow-up at a very general configuration $\mathbf{p}$. Define $M_\sigma : N^1(X) \to N^1(X)$ and $\check{M}_\sigma : N_1(X) \to N_1(X)$ by

$$M_\sigma = \left(\begin{array}{c|c} M & 0 \\ \hline 0 & I_5 \end{array}\right)\left(\begin{array}{c|c} 1 & 0 \\ \hline 0 & \Pi_\sigma \end{array}\right), \quad \check{M}_\sigma = \left(\begin{array}{c|c} \check{M} & 0 \\ \hline 0 & I_5 \end{array}\right)\left(\begin{array}{c|c} 1 & 0 \\ \hline 0 & \Pi_\sigma \end{array}\right),$$

where Π_σ is the permutation matrix for σ. The class of the strict transform of a divisor D under $\overline{\mathrm{Cr}}_{I_n} \circ \cdots \circ \overline{\mathrm{Cr}}_{I_1}$ is $\left(\begin{smallmatrix} 1 & 0 \\ 0 & \Pi_\sigma \end{smallmatrix}\right)^{-n} M_\sigma^n([D])$. Since D is movable and each $\overline{\mathrm{Cr}}_{I_j}$ is an isomorphism in codimension 1, this strict transform is a movable divisor as well. This strict transform is a divisor on a different blow-up $X_{\mathbf{q}}$ (as in Section 3), but if $\mathbf{p}$ is very general then by Lemma 2.2 this defines a movable class on X as well, and so $M_\sigma(\overline{\mathrm{Mov}}(X)) = \overline{\mathrm{Mov}}(X)$. Thus $M_\sigma : N^1(X) \to N^1(X)$ is a linear map which preserves the effective and movable cones. Similarly, if C is a curve with strict transforms disjoint from the indeterminacy loci of each $\overline{\mathrm{Cr}}_k$, its strict transform has class $\left(\begin{smallmatrix} 1 & 0 \\ 0 & \Pi_\sigma \end{smallmatrix}\right)^{-n} \check{M}_\sigma^n([C])$ by Corollary 4.2. The following lemma summarizes the essential properties of M_σ.

Lemma 4.4. *The linear transformation* M_σ *has characteristic polynomial* $p(t) = (t+1)(t-1)\, t^4 q(t+t^{-1})$*, where* $q(t) = t^4 - 3t^3 + 4t - 1$*.* M_σ *has four real eigenvalues:* 1*,* -1*,* $\lambda \approx 1.800$ *and* $1/\lambda$*. When the* λ*-eigenvector* D_λ *is written as* $H - \sum r_i E_i$*, the first two coefficients satisfy* $r_1 + r_2 > 1$*.*

Proof. The claims about the eigenvalues are easily verified from the characteristic polynomial. To obtain the claimed inequality on the coefficients, one may express the components of the eigenvector in terms of the dominant eigenvalue and compute their approximate values. □

Remark 4.5. To three decimal places, D_λ is given in components by

$$D_\lambda \approx (1, -0.640, -0.634, -0.615, -0.554, -0.355, -0.352, -0.341, -0.307, -0.197).$$

Remark 4.6. As in the two-dimensional case of Remark 3.5, the matrix $\left(\begin{smallmatrix} M & 0 \\ 0 & I_{k-4} \end{smallmatrix}\right)$ and permutation matrices generate the action of a Coxeter group of type $T_{2,4,5}$ on $N^1(X_{\mathbf{p}})$. The eigenvectors of many elements other than the M_σ considered above have similar properties, including a non-closed diminished base locus.

5. The geometry of D_λ

Lemma 5.1. *The class* D_λ *lies in* $\overline{\mathrm{Mov}}(X)$ *and spans an extremal ray on* $\overline{\mathrm{Eff}}(X)$*.*

Proof. The two claims follow from Lemma 2.1 by taking $V = N^1(X)$ and $T = M_\sigma$, with $G = \overline{\mathrm{Mov}}(X)$ and $G = \overline{\mathrm{Eff}}(X)$ respectively. The hypothesis on the dominant eigenvalue is verified in Lemma 4.4. □

Lemma 5.2 (= Theorem 1.1, (i), (ii))**.** *If* $\mathbf{p}$ *is very general, there is an infinite set of curves* $C_n \subset X = X_{\mathbf{p}}$ *such that* $D_\lambda \cdot C_n < 0$*, and* $\mathbf{B}_-(D_\lambda)$ *is not closed. The curves* C_n *are Zariski dense on* X*.*

Proof. The strategy is to construct curves C_n in the classes $\check{M}_\sigma^n([C_0])$, where C_0 is a line through p_1 and p_2. By Theorem 4.3, we can find a sequence of configurations $\mathbf{p}_j$, defined for all integers j, with $\mathbf{p}_0 = \mathbf{p}$ and such that the maps $\overline{\mathrm{Cr}}_{I_j} : X_{\mathbf{p}_{j-1}} \dashrightarrow X_{\mathbf{p}_j}$ are defined for all j. We may additionally assume that if $\ell \subset X_{\mathbf{p}_j}$ is a line not contained in the indeterminacy locus of $\overline{\mathrm{Cr}}_{I_{j+1}}$, then for all $k \geq 0$ the strict transform of ℓ on $X_{\mathbf{p}_{j+k}}$ is disjoint from the indeterminacy locus of $\overline{\mathrm{Cr}}_{j+k+1}$.

Suppose that $\bar{\ell} \subset X_{\mathbf{p}_{-n}}$ is the strict transform of a line between p_i and p_j. By Theorem 4.3, as long as p_i and p_j are not among the base points of Cr_{I_1}, the composition $\overline{\mathrm{Cr}}_{I_n} \circ \cdots \circ \overline{\mathrm{Cr}}_{I_1}$ is well-defined for all n, and the strict transforms of $\bar{\ell}$ are disjoint from the indeterminacy loci of the maps $\overline{\mathrm{Cr}}_{I_j}$. Taking $\bar{\ell}$ to be the line between $p_{\sigma^n(1)}$ and $p_{\sigma^n(2)}$ on $X_{\mathbf{p}_{-n}}$, we thus obtain a curve $C_n \subset X$ with class $\check{M}_\sigma^n([C_0])$, where $[C_0] = h - e_1 - e_2$ is the class of a line through the first two points. Note that $\overline{\mathrm{Cr}}_{I_{-n+1}} : X_{\mathbf{p}_{-n}} \dashrightarrow X_{\mathbf{p}_{-n+1}}$ is centered at $p_{\sigma^{n-1}(1)}, \ldots, p_{\sigma^{n-1}(4)}$. Since $\sigma^n(1) = \sigma^{n-1}(5)$ and $\sigma^n(2) = \sigma^{n-1}(6)$, ℓ is not among the curves in the indeterminacy locus of $\overline{\mathrm{Cr}}_{I_{-n+1}}$.

The computation of D_λ in Lemma 4.4 gives $D_\lambda \cdot C_0 = 1 - (r_1 + r_2) < 0$, and so

$$D_\lambda \cdot C_n = (\lambda^{-n} M_\sigma^n D_\lambda) \cdot (\check{M}_\sigma^n C_0) = \lambda^{-n}(D_\lambda \cdot C_0) < 0.$$

By (3) of Lemma 2.3, each curve C_n is contained in $\mathbf{B}_-(D_\lambda)$. However, D_λ is movable and so $\mathbf{B}_-(D_\lambda)$ contains no divisors by (7) of the same lemma. It follows that $\mathbf{B}_-(D_\lambda)$ is a countable union of curves.

We now show that the curves are Zariski dense. Suppose that $S \subset X$ is any surface, and let $\psi : \tilde{S} \to X$ be the inclusion of a resolution of S. Since D_λ is movable, S is not contained in the base locus of $D_\lambda + A$ for any ample A, and thus $\psi^*(D_\lambda)$ is pseudoeffective. If $C_n \subset S$, then a curve $\bar{C}_n \subset \tilde{S}$ mapping finitely to C_n has $(\psi^*(D_\lambda) \cdot \bar{C}_n)_{\tilde{S}} = (D_\lambda \cdot C_n)_X < 0$. However, a pseudoeffective $\mathbb{R}$-divisor on a smooth surface can have negative intersection with only finitely many curves, namely those in the support of the negative part of its Zariski decomposition (recalled in Theorem 6.1). Thus only finitely many of the curves C_n are contained in any surface. □

The first few classes $[C_n] = \delta h - \sum_i \mu_i e_i$ are given below.

n	δ	μ_1	μ_2	μ_3	μ_4	μ_5	μ_6	μ_7	μ_8	μ_9
0	1	1	1	0	0	0	0	0	0	0
1	3	1	1	1	1	1	1	0	0	0
2	7	3	2	2	2	1	1	1	1	1
3	13	4	4	4	4	3	2	2	2	1
4	25	8	8	8	7	4	4	4	4	3
5	45	14	14	14	13	8	8	8	7	4

On a given variety X, the set of divisors for which $\mathbf{B}_-(D)$ is not closed has measure 0 in $N^1(X)$; all such classes are *unstable* in the sense of [8]. Nevertheless, one expects that on "sufficiently complicated" varieties there should exist divisors for which $\mathbf{B}_-(D)$ is not closed. The following gives one result in this direction.

Corollary 5.3. *Suppose that Y is a normal projective threefold. There exists a finite set of points $q_1, \dots, q_j$ on Y such that if $r : Y' \to Y$ is the blow-up of the q_i, there is an $\mathbb{R}$-divisor D on Y' for which $\mathbf{B}_-(D)$ is not closed.*

Proof. Fix a separable finite map $s : Y \to \mathbb{P}^3$, and let $\mathbf{p}$ be a very general set of 9 points in $\mathbb{P}^3$, none of which is contained in the branch locus of s. Take the q_i to be the preimages of these 9 points under s, so there is a map $s' : Y' \to X_{\mathbf{p}}$. If D_λ is the divisor of the previous theorem, then $s'^* D_\lambda$ is a movable divisor, which has negative intersections with the preimages of each of the curves C_n. As above, it follows that $\mathbf{B}_-(s'^* D_\lambda)$ is a countable union of curves. □

Though the divisor D_λ is not big, a standard construction gives a big $\mathbb{R}$-divisor on a smooth 4-fold with non-closed diminished base locus. Fix an embedding $X \to \mathbb{P}^N$, let $CX \subset \mathbb{P}^{N+1}$ be the projective cone over X, and take $p : Y \to CX$ the blow-up at the cone point. The map p is birational with a unique exceptional divisor $E \cong X$; write $i_E : X \to Y$ for the inclusion. The variety Y has the structure of a $\mathbb{P}^1$-bundle $q : Y \cong \mathbb{P}_X(\mathcal{O}_X \oplus \mathcal{O}_X(1)) \to X$.

Lemma 5.4. *There exists a big $\mathbb{R}$-divisor D'_λ on Y with $\mathbf{B}_-(D'_\lambda)$ a countable union of curves.*

Proof. Let H be an ample divisor on CX with support disjoint from the cone point, and set $D'_\lambda = p^* H + q^* D_\lambda$. Choosing H sufficiently large, we may assume the base locus of D'_λ is contained in E. Observe that D'_λ is the sum of a big divisor and a pseudoeffective one, and thus big.

Properties (2), (4), and (5) of Lemma 2.3 imply that $\mathbf{B}_-(D'_\lambda) \subseteq \mathbf{B}_-(p^* H) \cup \mathbf{B}_-(q^* D_\lambda) = \mathbf{B}_-(q^* D_\lambda) = q^{-1}\mathbf{B}_-(D_\lambda)$. Furthermore, the choice of H implies that $\mathbf{B}_-(D'_\lambda) \subseteq E$, and so $\mathbf{B}_-(D'_\lambda) \subseteq q^{-1}\mathbf{B}_-(D_\lambda) \cap E$, which is a countable union of curves. Moreover, each curve $C'_j = i_E(C_j)$ has $C'_j \cdot D'_\lambda = q(C'_j) \cdot D_\lambda < 0$, and so $C'_j \subset \mathbf{B}_-(D'_\lambda)$. It follows that $\mathbf{B}_-(D'_\lambda)$ is a countable union of curves, all contained in E. □

6. Zariski Decomposition of D_λ

The non-closedness of $\mathbf{B}_-(D_\lambda)$ further implies that D_λ admits no Zariski decomposition in several standard senses. Recall the form of decomposition in dimension two:

Theorem 6.1 (Zariski decomposition theorem, e.g. [18])**.** *Let D be a pseudoeffective $\mathbb{R}$-divisor on a smooth projective surface X. There exists an effective divisor $N = \sum_i a_i N_i$ such that $P = D - N$ is nef, $(N_i \cdot N_j)$ is negative definite, and $P \cdot N_i = 0$.*

There are several analogues of Zariski decompositions for divisors on higher-dimensional varieties, imposing conditions which ensure the retention of useful properties of the two-dimensional version. One decomposition which always exists and has proved important is the divisorial Zariski decomposition of a pseudoeffective $\mathbb{R}$-divisor D, due to Nakayama.

Definition 6.2 ([16])**.** Suppose that D is an $\mathbb{R}$-divisor. For a prime divisor E on X, let

$$\sigma_E(D) = \sup_{A \text{ ample}} \left(\min_{D' \equiv_{\text{num}} D + A} \operatorname{ord}_E(D') \right).$$

Set $N_\sigma(D) = \sum_E \sigma_E(D) \cdot E$, and $P_\sigma(D) = D - N_\sigma(D)$. This is a finite sum, and $P_\sigma(D) \in \overline{\operatorname{Mov}}(X)$. When D is a big $\mathbb{Q}$-divisor, in fact $\sigma_E(D) = \min_{D' \equiv_{\text{num}} D} \operatorname{ord}_E(D')$.

In dimension two, this coincides with the standard Zariski decomposition, but in higher dimensions $P_\sigma(D)$ is only movable and not in general nef. To obtain a closer analogue of the Zariski decomposition, given a pseudoeffective $\mathbb{R}$-divisor on a smooth variety X, one might ask for a birational modification $f : Y \to X$ and a decomposition $f^*D = P + N$, with P nef and N effective. This is termed a *weak Zariski decomposition* by Birkar [4]. One might additionally ask that:

(1) CKM: the maps $H^0(Y, \mathcal{O}_Y(\lfloor mP \rfloor)) \to H^0(Y, \mathcal{O}_Y(\lfloor mf^*D \rfloor))$ are all isomorphisms.
(2) Fujita: if $g : Y' \to Y$ is birational, and $P' \leq g^*f^*D$ is nef, then $P' \leq g^*P$.
(3) Nakayama: $P = P_\sigma(f^*D)$ is the positive part of the divisorial Zariski decomposition.

Each of these seeks to extend a property of the usual two-dimensional Zariski decomposition to the higher-dimensional setting. The survey [18] of Prokhorov introduces the important properties of these and other higher-dimensional versions of the Zariski decomposition. Nakayama constructed an example of an $\mathbb{R}$-divisor on a $\mathbb{P}^2$-bundle over an abelian surface which admits no Zariski decomposition any of these three senses [16]. However, the divisor of Nakayama's example is itself big, thus effective, and trivially admits a weak Zariski decomposition. However, we show D_λ does not admit a weak Zariski decomposition, and that D'_λ of Lemma 5.4 is another example of big divisor with no decomposition in the sense of Nakayama.

Lemma 6.3 (= Theorem 1.1, (iii))**.** *D_λ does not admit a weak Zariski decomposition, and D'_λ does not admit a Zariski decomposition in the sense of Nakayama.*

Proof. Suppose that $f^*D_\lambda = P + N$ where N is effective. For each n, pick a curve $\tilde{C}_n$ on Y mapping finitely to C_n, and let $d_n = \deg(\tilde{C}_n \to C_n)$. Only finitely many of the $\tilde{C}_n$ are contained in $\operatorname{Supp} N$, since these curves are Zariski dense. On the other hand, for any curve $\tilde{C}_n$ not contained in $\operatorname{Supp} N$, we have $\tilde{C}_n \cdot N \geq 0$, and so compute $d_n(D_\lambda \cdot C_n) = f^*D_\lambda \cdot \tilde{C}_n = P \cdot \tilde{C}_n + N \cdot \tilde{C}_n \geq 0$, a contradiction. Similarly, the non-closedness of $\mathbf{B}_-(D'_\lambda)$ implies this divisor does not admit a Zariski decomposition in the sense of Nakayama [16, pg. 28]. ☐

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References

1. Turgay Bayraktar, *Green currents for meromorphic maps of compact Kähler manifolds*, Journal of Geometric Analysis (2012), 1–29.
2. Eric Bedford and Kyounghee Kim, *On the degree growth of birational mappings in higher dimension*, J. Geom. Anal. **14** (2004), no. 4, 567–596.
3. ______, *Periodicities in linear fractional recurrences: degree growth of birational surface maps*, Michigan Math. J. **54** (2006), no. 3, 647–670.
4. Caucher Birkar, *On existence of log minimal models and weak Zariski decompositions*, Math. Ann. **354** (2012), no. 2, 787–799.
5. Garrett Birkhoff, *Linear transformations with invariant cones*, Amer. Math. Monthly **74** (1967), 274–276.

6. Arthur B. Coble, *Algebraic geometry and theta functions*, American Mathematical Society Colloquium Publications, vol. 10, American Mathematical Society, Providence, R.I., 1982, Reprint of the 1929 edition.
7. Igor Dolgachev and David Ortland, *Point sets in projective spaces and theta functions*, Astérisque (1988), no. 165, 210 pp. (1989).
8. Lawrence Ein, Robert Lazarsfeld, Mircea Mustaţă, Michael Nakamaye, and Mihnea Popa, *Asymptotic invariants of base loci*, Ann. Inst. Fourier (Grenoble) **56** (2006), no. 6, 1701–1734.
9. Antonio Laface and Luca Ugaglia, *On a class of special linear systems of* $\mathbb{P}^3$, Trans. Amer. Math. Soc. **358** (2006), no. 12, 5485–5500.
10. ———, *Elementary* (-1)*-curves of* $\mathbb{P}^3$, Comm. Algebra **35** (2007), no. 1, 313–324.
11. Adrian Langer, *On positivity and semistability of vector bundles in finite and mixed characteristics*, J. of the Ramanujan Math. Soc. **28A** (2013), 287–309.
12. Robert Lazarsfeld, *Positivity in algebraic geometry. I*, Springer-Verlag, Berlin, 2004.
13. Curtis T. McMullen, *Dynamics on blowups of the projective plane*, Publ. Math. Inst. Hautes Études Sci. (2007), no. 105, 49–89.
14. Shigeru Mukai, *Counterexample to Hilbert's fourteenth problem for the 3-dimensional additive group*, RIMS Kyoto preprint **1343** (2001).
15. Masayoshi Nagata, *On rational surfaces. II*, Mem. Coll. Sci. Univ. Kyoto Ser. A Math. **33** (1960/1961), 271–293.
16. Noboru Nakayama, *Zariski-decomposition and abundance*, MSJ Memoirs, vol. 14, Mathematical Society of Japan, Tokyo, 2004.
17. Fabio Perroni and De-Qi Zhang, *Pseudo-automorphisms of positive entropy on the blowups of products of projective spaces*, Math. Ann. (2013), 1–21.
18. Yu. G. Prokhorov, *On the Zariski decomposition problem*, Tr. Mat. Inst. Steklova **240** (2003), no. Biratsion. Geom. Linein. Sist. Konechno Porozhdennye Algebry, 43–72.

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